MODULE-1

- Liquid drop model
- Bethe-Weizsacker Mass Formula
- Applications of Semi empirical mass formula
- Neutron Star

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NUCLEAR MODELS (L1)

What is an Atomic Nucleus?

Atoms : $10^{-10} m$ (Anstrom) ;   Nucleus : $10^{-15} m$ (Fermi)

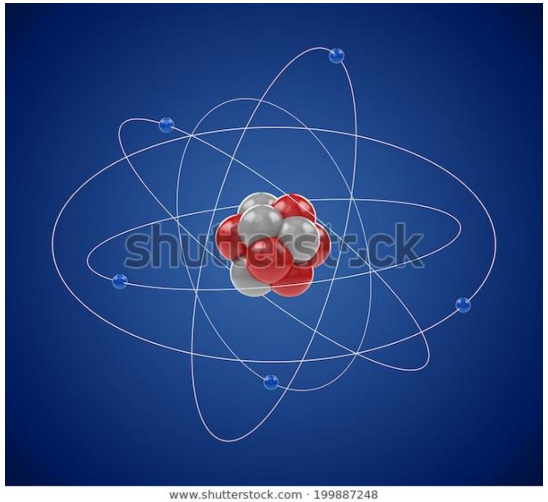
Atoms → Nucleus → Nucleons (protons & neutrons)

What are the constituents of a Nucleus?

Protons and Neutrons

Prout's Theory :

Nucleus is made up of protons and electrons. (Possibly, In those days electrons and protons were only discovered!)



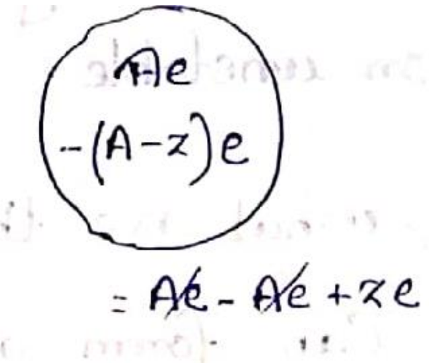
Is this proposition correct???

Nucleus comprises of A - protons and (A-Z) - electrons

with Z - electrons revolving around the nucleus.

A : Mass number

Z : Proton number



Explanation 1:

Uncertainty Principle

Can electron exist within a nucleus?

Ans: Uncertainty Principle:

$$\Delta x \Delta p \approx \frac{h}{2\pi}$$

$$\Delta p \approx \frac{6.626 \times 10^{-34}}{6.28 \times 10^{-14}} = 1.05 \times 10^{-20} Kg.m/s$$

Considering an ultra-relativistic electron with momentum of this order of magnitude we obtain,

$$E \approx \mathrm{p} \times c \approx \Delta p \times c \approx 20\ MeV$$

$$1eV = 1.6 \times 10^{-19} Joule \text{ and } m = 9.1 \times 10^{-31} Kg \approx 0.5\ MeV$$

Note that $1\ MeV = 10^6 eV$; $1\ GeV = 10^9 eV$; $1TeV = 10^{12} eV$

Electrons coming out of nucleus are found to have energy (3-5)MeV.

Conclusion : Electrons with such huge amount of energy can not exist within a nucleus. Hence, they can not be the constituent of a nucleus.

Mass of proton $= m_p = 938.3\ MeV$

Mass of neutron $= m_n = 939.5\ MeV$ (Why???)

Explanation 2 :

Statistics:

Consider ${}^{14}_{7}N$: No. of protons =14; No of electrons = 7 ; No of protons and electrons = 21

Protons and electrons both are spin ½ fermions, hence satisfy Fermi-Dirac statistics.

The nucleus being comprised of 14+7=21 spin ½ fermions, must satisfy FD statistics. But experimental observations show that the nucleus satisfies BE statistics. Why????

Explanation 3:

Magnetic moment:

Atomic Magnetic Moment : $\mu_B = \frac{e\hbar}{2m_e}$ (Bohr Magneton)

Nuclear Magnetic Moment : $\mu_N = \frac{e\hbar}{2m_p}$ (Nuclear Magneton)

If nucleus would have made up of electrons and protons $\mu_B \gg \mu_N$

Experimental observations showed that measured magnetic moments of nuclei are of the order μ_N (at least 3 order of magnitude smaller than that of atomic counterpart.

Conclusion: Nucleus is made up of PROTONS & NEUTRONS (Fermions)

What are the combinations of neutrons and protons leading to stable nucleus?

Depending upon the measured life time are considered as ‘Stable’ or ‘Unstable’.

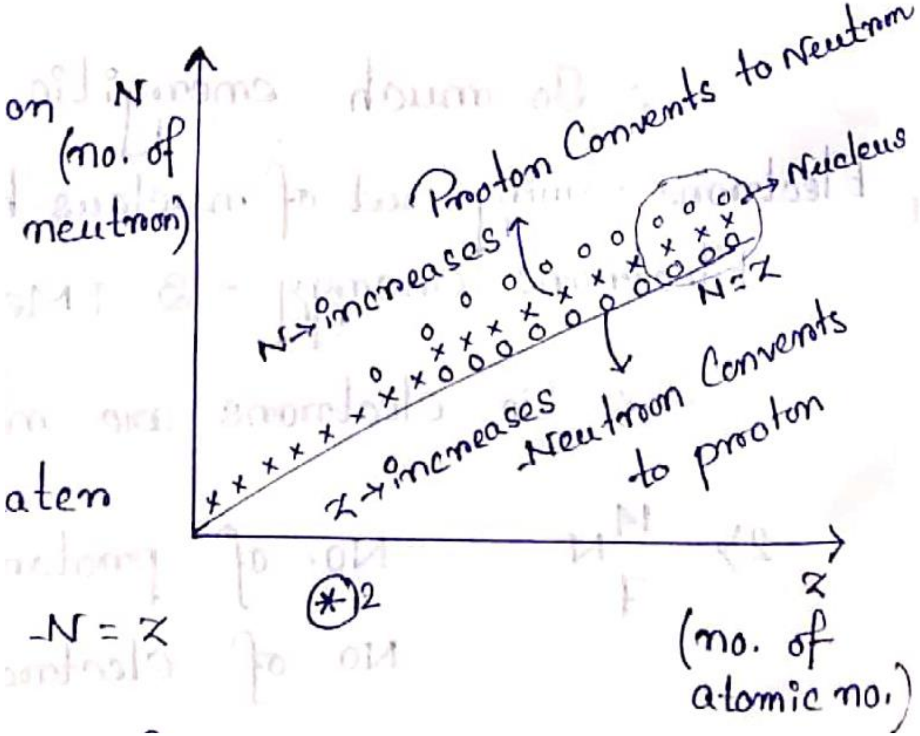
Why can’t we form a nucleus made up of only protons or only neutrons?

Only certain combination of p-n can exist in nature.

Due to Coulomb repulsion a nucleus can not be comprised of protons only.

What holds nucleons inside the nucleus? Some force must be there!!

- $A = Z + N$ (N= neutron number)
- $\times$ marks denote stable nuclei and o denotes unstable nuclei.
- No nuclei can exist below the solid line representing $Z = N$ line.
- Low A values corresponds to $Z = N$
- Intermediate A corresponds to unstable nuclei which convert into stable one by inter-conversion of neutron-proton.
- For large A, Nuclear Fission occurs.
- There must be a guiding principle which dictate us what combinations are allowed.



- Stability of a nucleus is determined by $\frac{N}{Z}$ ratio of a particular nucleus.
- Unstable nuclei have shorter life time.

Binding Energy (BE) of a Nucleus

$$E_B = \frac{A(A-1)}{2} \approx \frac{A^2}{2}$$

$\frac{E_B}{A} \propto A$ (Long range force)

Each nucleon is surrounded by ν no of nucleons

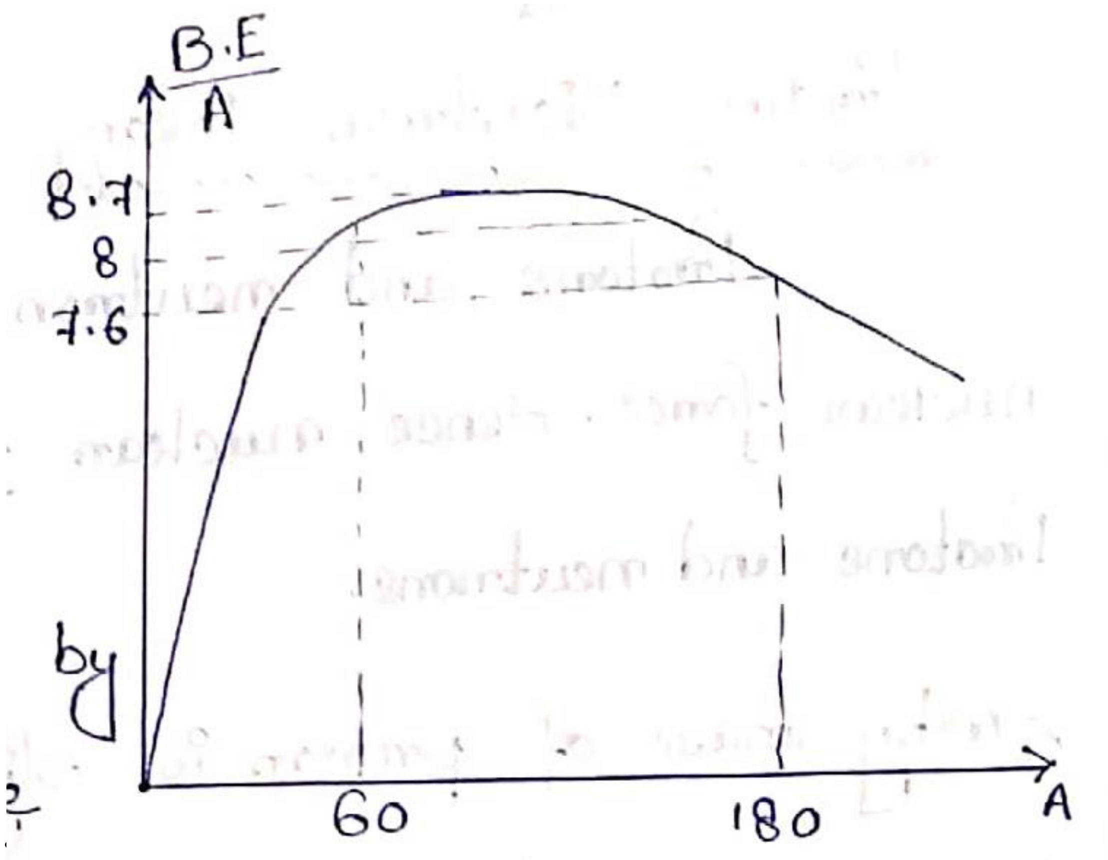
Total no of pairs contributing to the nuclear force is $\frac{\nu A}{2}$

½ factor is considered to avoid double counting.

Each pair contributes ϵ amou nt of energy.

Hence the BE of the nucleus is : $E_B = \frac{\nu A}{2}\epsilon$

Therefore, $\frac{E_B}{A} = \frac{\nu\epsilon}{2} =$ constant (Short range force)



- The BE curve shows a flat region over a wide range of A.
- Consistent with short range nature of nuclear force., i.e., each nucleon can interact with its immediate neighbours only.

Proton-Neutron Theory:

Protons and neutrons both can feel strong force (nuclear force). Hence nuclear force can not distinguish them leading to charge independence nature of nuclear force.

Protons and neutrons are almost mass degenerate. Known as Nucleons, belong to same isospin doublet.

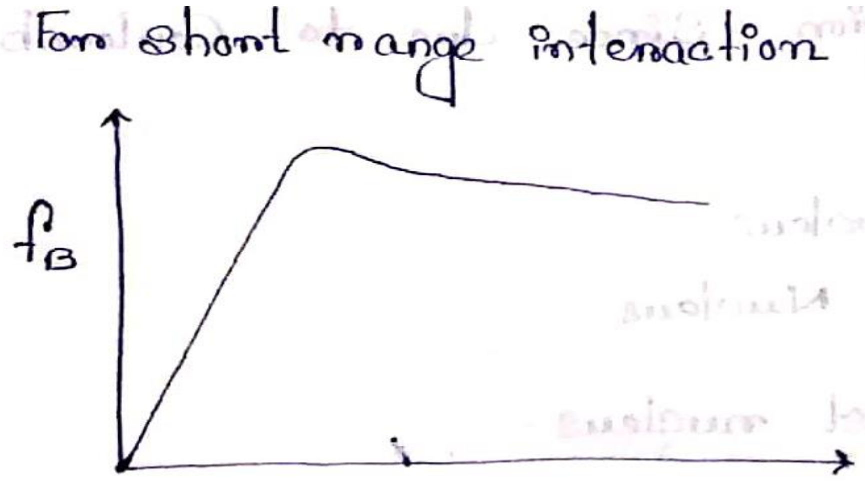
(U)NUCLEAR PHYSICS

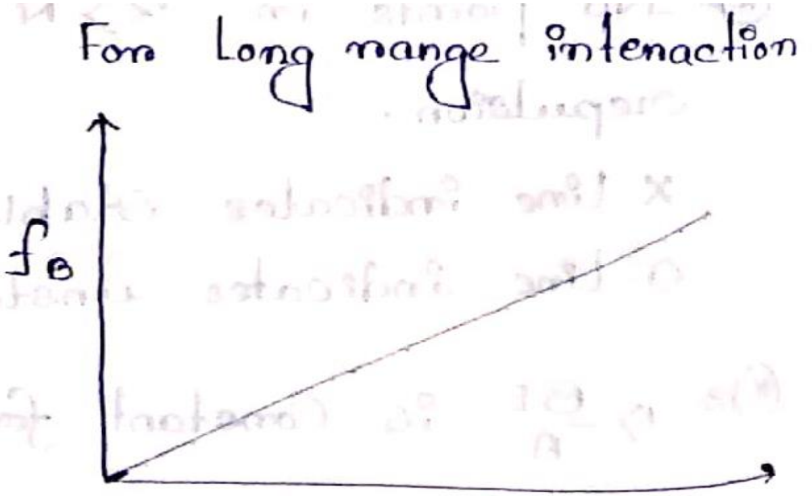
Nuclear Physics is not like Atomic Physics

The nature of force is still unknown!

To propose various models for explaining the features is the only way to know about the subject.

Different models consider different forms of interaction of nucleons.





Binding Energy (BE):

BE is defined as the minimum amount of energy required to pull out one nucleon (proton/neutron) from the nucleus and to make it isolated from the rest.

Consider an atomic nucleus: ${}^{A}_{Z}X$ with $N = A - Z$

$m({}^{A}_{Z}X)c^2$ is the mass of the nucleus.

Atomic masses can be measured with great precision compared to nuclear masses. Hence, we try to express nuclear masses in terms of atomic masses.

$$m({}^{A}_{Z}X)c^2 + BE = Zm_pc^2 + Nm_nc^2$$

$$m({}^{A}_{Z}X)c^2 + Zm_ec^2 = (Zm_p + Zm_e)c^2 + Nm_nc^2 - BE$$

$$m^{At}({}^{A}_{Z}X)c^2 + BE^{At}({}^{A}_{Z}X) = Zm_Hc^2 + Nm_nc^2 + Z\,BE^{At}({}^{1}_{1}H) - BE$$

$$m^{At}({}^{A}_{Z}X)c^2 = Zm_Hc^2 + Nm_nc^2 + Z\,BE^{At}({}^{1}_{1}H) - BE^{At}({}^{A}_{Z}X) - BE$$

Comparison of BE of an atom and a nucleus:

Hydrogen atom:

BE of H-atom = -13.6 eV ≈ 10 eV

$m_p \approx 1000$ MeV $=10^9$eV

Therefore, 1 part in 10^8

He atom

${}_2^4He$ nucleus consists of 2p + 2n

Mass = 4000 MeV

BE of He = 28 MeV

Therefore, BE is 1 part in 100

Hence, we can neglect the difference in atomic BE $\left(\left\{Z\,BE^{At}({}_1^1H) - BE^{At}({}_Z^AX)\right\}\right)$ w.r.t nuclear BE

Hence the expression for BE is

$$m^{At}\left({}_Z^AX\right)c^2 = Zm_Hc^2 + Nm_nc^2 - BE$$

By knowing the atomic mass, we can estimate nuclear BE.

NUCLEAR MODELS(L2)

How do we find an expression for BE of nucleus?

Problem: Ignorance about the exact form of Nuclear force

Solution: Based on i) Experimental inputs

ii) Classical concepts

iii) Quantum concepts

An expression for BE can be obtained considering Liquid Drop Model

LIQUID DROP MODEL

Proposed by Bohr-Wheeler& Frankel in 1937

Similarities between a drop of liquid and nucleus:

1. Constant density

$$\rho_m = \frac{A}{\frac{4}{3}\pi R^3} \qquad \text{with } R = R_0 A^{\frac{1}{3}}$$

$$\rho_m = \frac{1}{\frac{4}{3}\pi R_0^3} = \text{constant}$$

Nuclear matter density is independent of nature of nucleus.(independent of A)

Both liquid drop and nucleus possess constant density.

2. Short range force:

Like molecules of liquid drop nucleons within the nucleus interacts with its immediate surroundings only.

3. Evaporation:

Emission of α, β, γ is equivalent to evaporation process.

BE is equivalent to latent heat of vaporization

4. Formation of compound nucleus is equivalent to condensation of liquid drop.
5. Nuclear fission is equivalent to splitting of liquid drop into two parts.
6. Nuclear potential among all nucleons ($V_{pp} \approx V_{nn} \approx V_{pn}$) is equivalent to interaction among all liquid molecules via same potential
7. BE of a nucleus equivalent to Surface tension of liquid drop.

Terms present in BE expression:

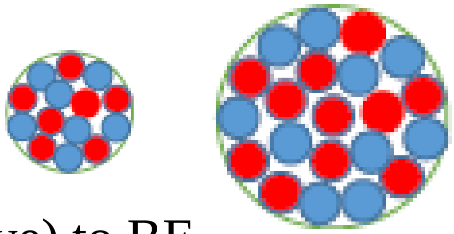
1) Volume Energy
2) Surface Energy
3) Coulomb Energy

Classical origin

4) Asymmetry Energy
5) Pairing Energy

Quantum origin

Semi-empirical mass formula: (Bethe-Weizsacker)



1. Volume Energy: As the volume of a nucleus increases more and more nucleons can be accommodated. Hence nucleons are more tightly bound with each other inside the nucleus resulting in increase in BE of the nucleus. This contribution (+ ve) to BE is termed as Volume Energy term.

   $E_V = a_V A$; a_V =volume energy co-efficient

2. Surface Energy: Nucleus is considered as spherical object. As the surface area increases, more number of nucleons can reside on the surface of the nucleus. These nucleons do not have further nucleons on the outer surface to be bounded with. As a results they can easily be pulled out thereby deceasing in BE of the nucleus. More the nucleus is bigger, BE would be smaller. Therefore, the surface energy term is proportional to the surface area of the corresponding nucleus. This contribution (- ve) to BE is termed as Surface Energy term.

   $E_s \propto 4\pi R^2 = -a_S A^{\frac{2}{3}}$; a_s = surface energy co-efficient, R =radius of the nucleus

3. Coulomb Energy: Nucleus is considered to be a uniformly charged sphere. Protons are assumed to be uniformly distributed inside the nucleus. This assumption is valid as nucleus is a densely packed object. The repulsive force between protons tend to weaken the nuclear binding. This repulsive force being long range by nature, the number of proton pairs contributing in this energy is $\frac{1}{2}Z(Z-1) \approx \frac{1}{2}Z^2$ for large Z.

We know that the electrostatic energy stored in solid sphere of radius R and carrying total charge Q is given by $E = -\frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$. '–ve' sign indicates that this would reduce the energy of the system.

Assuming the total nuclear charge, $Q = +Ze$ to be distributed uniformly inside the nucleus, the Coulomb energy stored in the nucleus is $E_C = -\frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R} = -\frac{3e^2}{20\pi\epsilon_0}\frac{Z^2}{R_0 A^{\frac{1}{3}}} = -a_C\frac{Z^2}{A^{\frac{1}{3}}}$ where, $a_C = \frac{3e^2}{20\pi\epsilon_0 R_0}$

Thus we see that the Coulomb Energy contribution reduces the BE of nucleus.

Problem: Derive the expression for Coulomb energy and estimate the value of a_C in MeV unit.

$$a_c = \frac{1}{4\pi\epsilon_0}\cdot\frac{3}{5}\frac{e^2}{R_0} = \frac{3}{5}\cdot\frac{(1.6\times10^{-19})^2}{4\pi(10^{-9}/36\pi)\times1.22\times10^{-15}}$$

$$= 0.00076\,u$$

$$= 0.71\,\text{MeV}$$

4. Asymmetry Energy: This term originates due to inequality of neutron number (N) and proton number (Z) in a nucleus. Quantum physics is the origin of this term and it reduces the BE of the nucleus. Here, a nucleus is considered as a many body problem because it consists of many protons and neutrons which are known as nucleons .

Assumptions:

- Any nucleon in a nucleus having A nucleons would experience a resultant average potential produced by the remaining $(A-1)$ nucleons.
- Energy levels are discrete and equispaced.**
- Nucleons obey Pauli's exclusion principle. Hence, each energy level can accommodate maximum 2 protons and 2 neutrons
- $N-Z$ plot shows that stable nuclei must satisfy $Z \leq N$. Equality holds for low A nuclei and N becomes greater than Z for large A

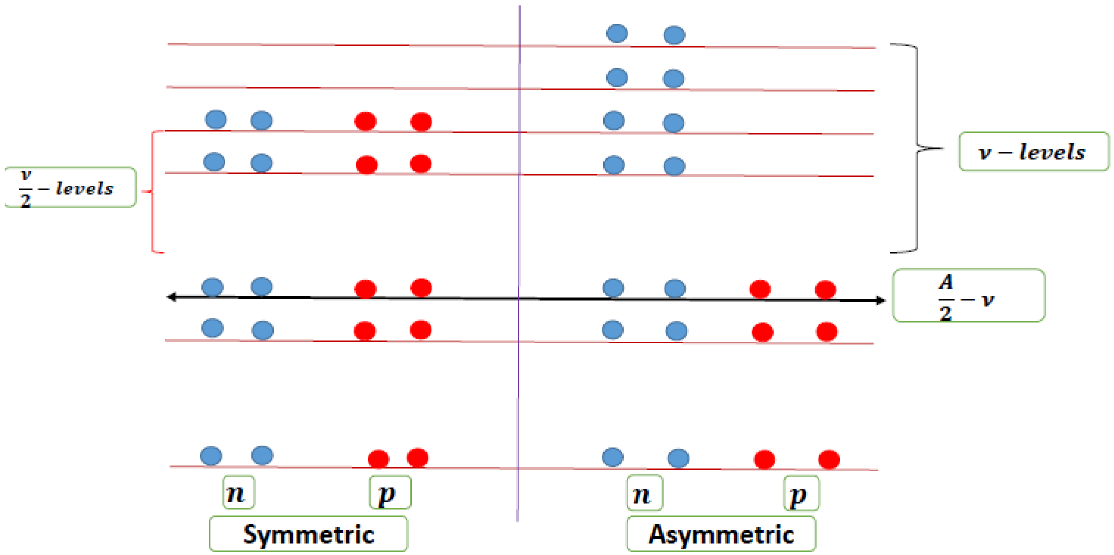
Consider a symmetric nucleus having $N = Z = \frac{A}{2}$

and an asymmetric nucleus with $Z = \frac{A}{2} - \nu$; $N = \frac{A}{2} + \nu$

and $N - Z = 2\nu$.

Assume the asymmetric nucleus to be comprised of $\left(\frac{A}{2} - \nu\right)$ no of protons and $\left(\frac{A}{2} + \nu\right)$ no of neutrons.

The black double-arrowed line denotes the energy level upto which $\left(\frac{A}{2} - \nu\right)$ no of protons can be filled up starting from the ground state subject to Pauli's exclusion principle. (see the adjacent figure) .



For the symmetric one there will be ν no of neutrons and ν no of protons left out for filling up the energy levels located above the black double-arrowed line. Each energy level can accommodate 2 protons and 2 neutrons. Therefore, $\frac{\nu}{2}$ no of energy levels will be filled up by ν no of neutrons and ν no of protons. Hence the energy of the symmetric nucleus be , E_1

$$E_1 = 4\delta + 2 \times 4\delta + 3 \times 4\delta + \ldots\ldots\ldots.\frac{\nu}{2} \text{ terms}$$

$$= 4\delta\left[1 + 2 + 3 + \ldots\ldots\ldots.\frac{\nu}{2} terms\right] = 4\delta\,\frac{\frac{\nu}{2}\left(\frac{\nu}{2} + 1\right)}{2} = \delta \times \nu\left(\frac{\nu}{2} + 1\right)$$

δ is the separation between two successive energy levels.

For the asymmetric nucleus there are 2ν no of neutrons in excess of the no of protons. Therefore, these neutrons will fill up ν no of energy levels located above the black double-arrowed line. (see the last figure).

The energy of the corresponding nucleus is E_2,

$$E_2 = 2\delta + 2 \times 2\delta + 3 \times 2\delta + \dots\dots\dots.\nu \text{ terms}$$

$$= 2\delta[1 + 2 + 3 + \dots\dots\dots.\nu \ terms] = 2\delta \frac{\nu(\nu+1)}{2} = \delta \times \nu(\nu+1)$$

The energy difference $E_2 - E_1$, is

$$E_2 - E_1 = \delta\left[\nu(\nu+1) - \nu\left(\frac{\nu}{2}+1\right)\right] = \delta\left[\frac{\nu^2}{2}\right] = \frac{(N-Z)^2}{8}\delta$$

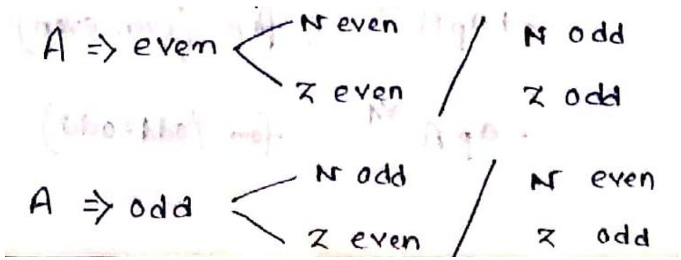
From the above expression the energy of the asymmetric nucleus is greater than that corresponding to symmetric one. As a result BE of the nucleus would decrease for the asymmetric case.

It has been observed that the highest nuclear Fermi energy level is almost fixed for all nuclei. Moreover, ground states of all nuclei are in generally the same. Hence the no of energy levels lying between these fixed two energy levels would increase when the no of nucleons increases. Consequently the energy gap between two successive energy levels decrease. Therefore, $\delta \propto \frac{1}{A}$

Hence the asymmetry energy term is , $E_a = -a_A \frac{(N-Z)^2}{A}$

'-ve' sign indicates that BE reduces due to presence of Asymmetry energy term

5. Pairing Energy: The origin of this term is Quantum Mechanical concepts. When two nucleons are paired up, their spins are oppositely oriented resulting in tightly bound state. Pairing lowers the energy of system. Therefore, more energy is required to pull out a particular type of nucleon from the nuclei having paired nucleons compared to the nuclei having unpaired nucleon.



S_N →Neutron Separation Energy → Energy required to separate one neutron from a nucleus.

S_P →Proton Separation Energy → Energy required to separate one proton from a nucleus.

For N =Even; S_N is large and N =Odd; S_N is low

For Z =Even; S_p is large and Z =Odd; S_p is low

- If N is even, it is difficult to take out one neutron from the nucleus because firstly we have to break the pair and then to pull it out more energy is needed. Therefore, S_N is larger for even N nuclei compared to the odd N nuclei.
- It has been observed that the no of β −stable nuclei for e-e nuclei is about 165 and the same for e-o nuclei is about 105. For o-o nuclei the corresponding no is only 4.
- $({}^{2}_{1}H, {}^{6}_{3}Li, {}^{10}_{5}B, {}^{14}_{7}N)$

Pairing Energy $E_P = \delta$

Empirically it is found that

$\delta = 0$ for $A = odd$ (e-o nuclei)

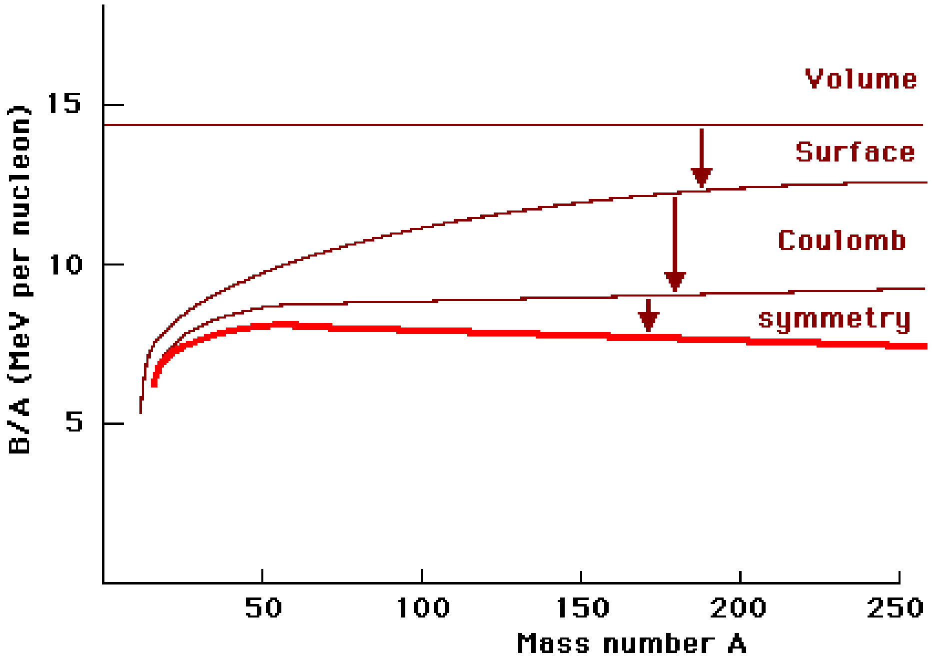
$= +a_P A^{-\frac{3}{4}}$ for $A =$ even (e-e nuclei)

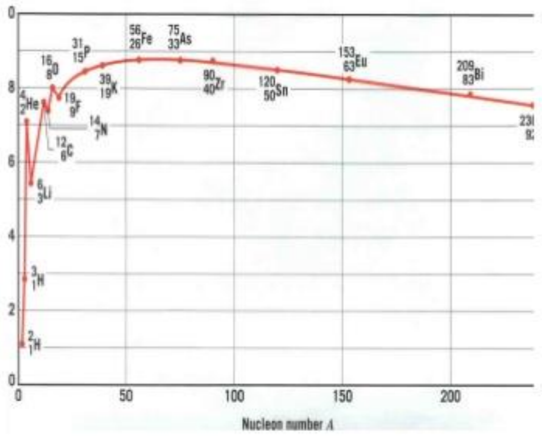
$= -a_P A^{-\frac{3}{4}}$ for $A =$even (o-o nuclei)

Therefore, the expression for BE is,

$$B = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(N-Z)^2}{A} + \delta$$

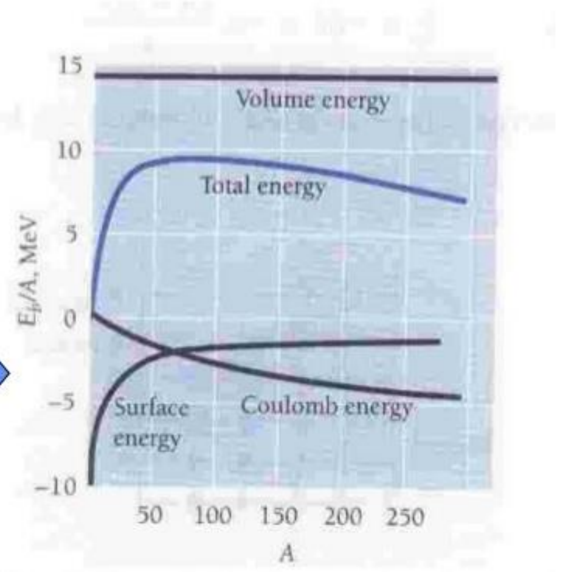
- The parameters of the BE expression are estimated from experimental inputs with data fitting technique. The values of such parameters are :
- $a_V = 15.8$ MeV
- $a_S = 17.6$ MeV
- $a_C =$0.72 MeV
- $a_A =$23.7 MeV
- $a_P =$33.5 MeV





Empirical binding energy per nucleon curve

theoretical binding energy per nucleon curve (using Liquid drop model concept)



It is called a semi-empirical mass formula because the formula has been derived obtaining inputs from theory as well as the observed results of experiments.

Semi : Theoretical concepts of CM & QM with physical concepts.

Empirical: Experimental inputs are used for determining the parameters: a_V, a_S, a_C, a_A, a_P

Bethe-Weizsacker Mass Formula (Semi-empirical mass formula)

$$B = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(N-Z)^2}{A} + \delta$$

$$M^{At}(A,Z)c^2 = ZM_H c^2 + Nm_n c^2 - B$$

$$M^{At}(A,Z)c^2 = ZM_H c^2 + Nm_n c^2 - a_V A + a_S A^{\frac{2}{3}} + a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_A \frac{(N-Z)^2}{A} - \delta$$

BE of He- nucleus:

Helium ${}^4_2He$ $(A = 4, Z = 2, N = 2)$

$m_H = 1.007825u$; $m_{He} = 4.00260u$; $m_n = 1.008665u$ & $1u$ =931.5MeV

$B = ZM_H c^2 + Nm_n c^2 - M^{At}({}^4_2He)c^2$ $\Rightarrow$ $B = 28.3$ MeV

$$B = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(N-Z)^2}{A} + \delta$$

e-e nucleus , $\delta = +a_P A^{-\frac{3}{4}}$, substituting parameters we obtain $B = 29.7$ MeV (Close agreement)

Applications of semi-empirical BE formula:

- $\alpha -$ decay
- Stability of a Nucleus (Mass parabola)
- $\beta -$ disintegration energy of mirror nuclei
- Formation of Neutron Stars

Explanation of $\alpha -$ decay from Semi-Empirical mass formula:

Spontaneous $\alpha - decay$ phenomena can be explained with Semi-Empirical mass formula. There is a lower bound on mass number, A of a particular nucleus for which this phenomena can occur.

$\alpha - decay:$

$$ {}_{Z}^{A}X \rightarrow {}_{Z-2}^{A-4}Y + {}_{2}^{4}\alpha $$

$Q =$ BE of daughter nuclei$-$BE of mother nucleus

$Q_\alpha = B(A-4, Z-2) + B_\alpha - B(A, Z)$ where, $B_\alpha = 28.3$ MeV

For spontaneous $\alpha - decay,$ $Q_\alpha \geq 0$

$$Q_\alpha = B(A-4, Z-2) + B_\alpha - B(A,Z)$$

$$Q_\alpha = 28.3MeV + B(A-4, Z-2) - B(A, Z-2) - B(A,Z) + B(A, Z-2)$$

$$= 28.3MeV \; - 4\frac{\partial B}{\partial A} - 2\frac{\partial B}{\partial Z} \quad \text{[using definition of differentiation] ------------ (1)}$$

Neglecting the Pairing term we can write,

$$B = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A}$$

$$\frac{\partial B}{\partial A} = a_V - \frac{2}{3} a_S A^{-\frac{1}{3}} + \frac{1}{3} a_C \frac{Z(Z-1)}{A^{\frac{4}{3}}} - a_A \left[\frac{2(A-2Z)}{A} - \frac{(A-2Z)^2}{A^2}\right] \text{ ------------- (2)}$$

$$\frac{\partial B}{\partial Z} = -a_C \frac{(2Z-1)}{A^{\frac{1}{3}}} + a_A \left[\frac{4(A-2Z)}{A}\right] \text{ -------------- (3)}$$

Substituting Eqs. (2) & (3) in Eq. (1), we get,

$$Q_\alpha = 28.3MeV \; - 4a_V + \frac{8}{3} a_S A^{-\frac{1}{3}} - \frac{4}{3} a_C \frac{Z(Z-1)}{A^{\frac{4}{3}}} + 4a_A \left[\frac{2(A-2Z)}{A} - \frac{(A-2Z)^2}{A^2}\right] + 2a_C \frac{(2Z-1)}{A^{\frac{1}{3}}} - 2a_A \left[\frac{4(A-2Z)}{A}\right]$$

$$= 28.3MeV \; - 4a_V + \frac{8}{3} a_S A^{-\frac{1}{3}} - \frac{4}{3} a_C \frac{Z(Z-1)}{A^{\frac{4}{3}}} - 4a_A \left[\frac{(A-2Z)^2}{A^2}\right] + 2a_C \frac{(2Z-1)}{A^{\frac{1}{3}}}$$

$$= 28.3MeV \; - 4a_V + \frac{8}{3} a_S A^{-\frac{1}{3}} - 4a_A \left(1 - \frac{2Z}{A}\right)^2 + \frac{4a_C Z}{A^{\frac{1}{3}}}\left(1 - \frac{Z}{3A}\right) + \frac{4a_C Z}{3A^{\frac{4}{3}}} - \frac{2a_C}{A^{\frac{1}{3}}} \text{ ------------- (4)}$$

Neglecting the last two terms (red in colour), we obtain,

$$Q_\alpha = 28.3MeV\ - 4a_V + \frac{8}{3}a_S A^{-\frac{1}{3}} - 4a_A\left(1-\frac{2Z}{A}\right)^2 + \frac{4a_C Z}{A^{\frac{1}{3}}}\left(1-\frac{Z}{3A}\right) \text{---------------} \quad (5)$$

Since, $\frac{4a_C Z}{3A^{\frac{4}{3}}} \approx 0.08$ and $\frac{2a_C}{A^{\frac{1}{3}}} \approx 0.3$; $[A = 150]$

Therefore, can be neglected compared to the other terms in Eq. (4).

Plugging in the expression of $z \approx 0.41A$ and the values of a_V, a_S, a_C and a_A in Eq. (4) we obtain,

$$Q_\alpha = -38MeV + 46.9A^{-\frac{1}{3}} + 1.02A^{\frac{2}{3}}$$ (Verify!)

Putting $x = A^{\frac{1}{3}}$ and $Q_\alpha \geq 0$, we get $x^3 - 37.3x + 46 = 0 \Rightarrow x \approx 5.36$ (Verify online!)

$$A \geq 154$$

The energy release for $A \geq 154$ is very small, hence barrier penetration probability is very small. Actually spontaneous $\alpha - decay$ is observed in the region $A \geq 200$.

NUCLEAR MODELS(L3)

Applications of semi-empirical BE formula:

- $\alpha -$ decay
- Stability of a Nucleus (Mass parabola)
- $\beta -$ disintegration energy of mirror nuclei
- Nuclear Fission
- Formation of Neutron Stars

Stability of a Nucleus (Mass parabola)

For a fixed mass number (A), any combination of N and Z is not allowed to make a stable nucleus. We shall obtain an expression for Z in terms of A for a stable nucleus using the semi-empirical mass formula.

What is a mass parabola? Why is it called so?

$$M^{At}(A,Z)c^2 = ZM_H c^2 + Nm_n c^2 - B$$

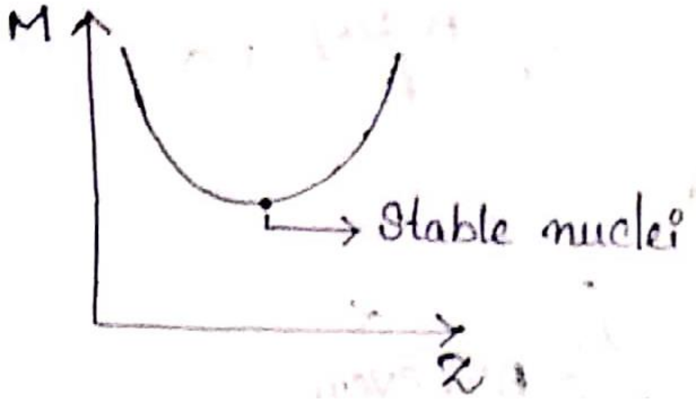
$$M^{At}(A,Z)c^2 = ZM_H c^2 + Nm_n c^2 - a_V A + a_S A^{\frac{2}{3}} + a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_A \frac{(N-Z)^2}{A} - \delta$$

This expression can be recast into the following form

$$M^{At}(A,Z)c^2 = BZ^2 + CZ + D$$

Represents an equation of parabola ($y = aZ^2 + bZ + cZ$) with mass of the atom as the axis of parabola.

$$M^{At}(A,Z)c^2 = BZ^2 + CZ + D$$



Where,

$$B = \frac{a_C}{A^{\frac{1}{3}}} + \frac{4a_A}{A}$$

$$C = M_H c^2 - m_n c^2 - \frac{a_C}{A^{\frac{1}{3}}} - 4a_A$$

$$D = Am_n c^2 - a_V A + a_S A^{\frac{2}{3}} + a_A A - \delta$$

For an extremum,

$$\frac{\partial M^{At}}{\partial Z} = 0$$

$$\Rightarrow 2BZ + C = 0 \Rightarrow Z_0 = -\frac{C}{2B}$$

This is the value of Z for which the corresponding nucleus becomes stable.

Show that $\left.\frac{\partial^2 M^{At}}{\partial Z^2}\right|_{Z_0=-\frac{C}{2B}} > 0$

For a stable atom the value of Z is

$$Z_0 = \frac{(m_n - M_H)c^2 + \frac{a_C}{A^{\frac{1}{3}}} + 4a_A}{\frac{2a_C}{A^{\frac{1}{3}}} + \frac{8a_A}{A}}$$

- $(m_n - M_H)c^2 = 0.78\ MeV$
- $\frac{a_C}{A^{\frac{1}{3}}} \ll 0.72$ for $A \approx 100$
- $4a_A \approx 100 MeV$

Approximation: Red coloured terms in the numerator may be neglected compared to blue cloured term.

Therefore, $Z_0 \approx \frac{4a_A A}{2\left[a_C A^{\frac{2}{3}} + 4a_A\right]} = \frac{A}{2}\frac{1}{\left[1+\frac{a_C}{4a_A}A^{\frac{2}{3}}\right]} = \frac{A}{2}\left[\frac{1}{1+0.0076\times A^{\frac{2}{3}}}\right] \approx \frac{A}{2}\left[1 - 0.0076 \times A^{\frac{2}{3}}\right]$

For lighter nuclei, $A = small;\ Z \approx \frac{A}{2}$

For heavier nuclei $Z = \frac{A}{2}\left[1 - 0.0076 \times A^{\frac{2}{3}}\right]$

For $A = 8;\ Z \approx 3.9\ (exact\ formula) \approx 4$ & for $A = 238;\ Z \approx 92.2(exact\ formula) \approx 92$

People assume $Z \approx (0.38 - 0.41)A$

$\beta-$ disintegration energy of mirror nuclei

Mirror Nuclei: Mirror nuclei are pairs of isobaric nuclei in which the proton numbers and neutron numbers are interchanged and differ by one unit. $(N-Z=1)$.

[Examples: $({}^{3}_{1}H, {}^{3}_{2}He), ({}^{7}_{3}Li, {}^{7}_{4}Be)\left({}^{11}_{5}B, {}^{11}_{6}C\right)\left({}^{15}_{7}N, {}^{15}_{8}O\right)$]

Nuclear Reaction : ${}^{A}_{Z}X \rightarrow {}_{Z-1}{}^{A}Y + \beta^{+} + \nu$ [Considering *odd A* nuclei]

$$M^{At}(A,Z)c^2 = ZM_Hc^2 + Nm_nc^2 - a_VA + a_S\, A^{\frac{2}{3}} + a_C\frac{Z^2}{A^{\frac{1}{3}}} + a_A\frac{(N-Z)^2}{A}$$

$$M^{At}(A,Z)c^2 = ZM_Hc^2 + (Z-1)m_nc^2 - a_VA + a_S\, A^{\frac{2}{3}} + a_C\frac{Z^2}{A^{\frac{1}{3}}} + a_A\frac{1}{A}$$

For daughter nucleus:

$$M^{At}(A,Z-1)c^2 = (Z-1)M_Hc^2 + Zm_nc^2 - a_VA + a_S\, A^{\frac{2}{3}} + a_C\frac{(Z-1)^2}{A^{\frac{1}{3}}} + a_A\frac{1}{A}$$

So we have,

$$M^{At}(A,Z)c^2 - M^{At}(A,Z-1)c^2 = \mathrm{M_H c^2 - m_n c^2} + a_C\frac{(2Z-1)}{A^{\frac{1}{3}}}$$

$$\therefore Q_{\beta^+} = M^{At}(A,Z)c^2 - M^{At}(A,Z-1)c^2 - 2m_ec^2$$

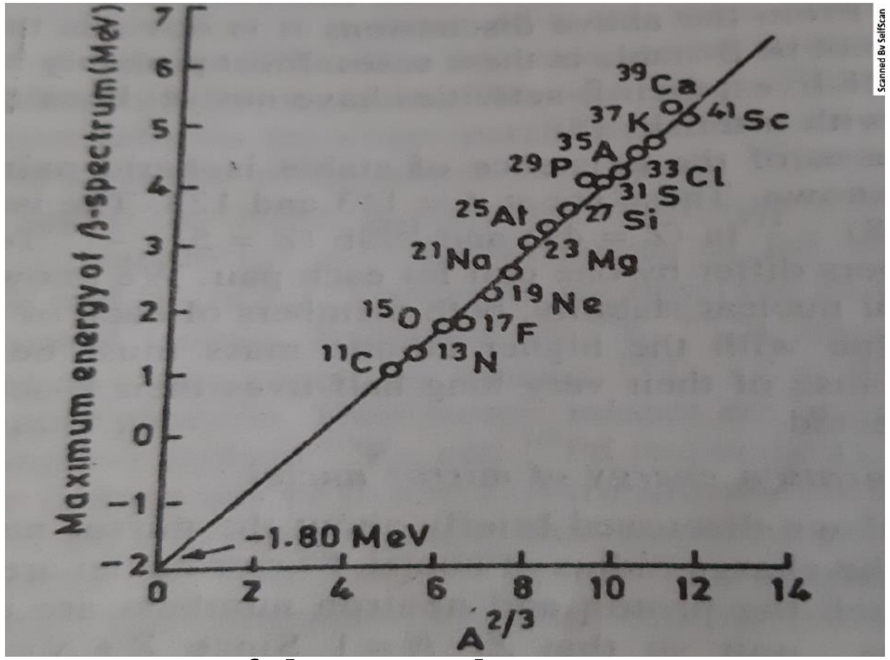
$$\therefore Q_{\beta^+} = M^{At}(A,Z)c^2 - M^{At}(A,Z-1)c^2 - 2m_e c^2$$

$$= M_H c^2 - m_n c^2 + a_C \frac{(2Z-1)}{A^{\frac{1}{3}}} - 2m_e c^2$$

$$= a_C\ A^{\frac{2}{3}} - (m_n - M_H + 2m_e)c^2$$

$$= a_C\ A^{\frac{2}{3}} - 1.8\ MeV$$

- The slope of the straight line plot of disintegration energy Q_{β^+} against $A^{\frac{2}{3}}$ can estimate the nuclear radius parameter R_0 provided a_C is known or vice e versa.
- The estimated radius parameter is on higher side because determination of coulomb energy includes some assumptions. When accurate estimation of Coulomb energy is made , the radius parameter is in better agreement with that determined by other methods.



- Coulomb energy : non-uniform nuclear charge distribution, requirement of discrete charge arrangement of charges on proton, effect of uncertainty in the localization of the protons, non-sphericity of the nucleus, correction of position of the protons.

Spontaneous Nuclear Fission

Nuclear fission is the phenomena where heavy nuclei split into nearly equal daughter nuclei. In may be induced by neutrons or it may occur spontaneously. A Nuclear fission reaction may be represented as :

$$ {}^{A}_{Z}X \rightarrow {}^{A_1}_{Z_1}X_1 + {}^{A_2}_{Z_2}X_2 $$

If we consider symmetric nucleus ($Z_1 = Z_2 = \frac{Z}{2}$ & $A_1 = A_2 = \frac{A}{2}$), the fission reaction may be represented as:

$$ {}^{A}_{Z}X \rightarrow {}^{\frac{A}{2}}_{\frac{Z}{2}}X_1 + {}^{\frac{A}{2}}_{\frac{Z}{2}}X_1 $$

Prompt neutrons are not considered here.

The above processes can occur if Q value of the reaction is positive.

$$ Q = M(A, Z) - M(A_1, Z_1) - M(A_2, Z_2) > 0 $$

For the symmetric case ,

$$ Q = M(A, Z) - 2 \times M\left(\frac{A}{2}, \frac{Z}{2}\right) > 0 $$

Atomic masses in terms of Semi-empirical mass formula (Neglecting pairing term)

$$ M(A, Z)c^2 = ZM_Hc^2 + Nm_nc^2 - a_V A + a_S A^{\frac{2}{3}} + a_C \frac{Z^2}{A^{\frac{1}{3}}} + a_A \frac{(A - 2Z)^2}{A} $$

Atomic masses in terms of Semi-empirical mass formula (Neglecting pairing term)

$$M(A,Z)c^2 = ZM_Hc^2 + Nm_nc^2 - a_V A + a_S A^{\frac{2}{3}} + a_C \frac{Z^2}{A^{\frac{1}{3}}} + a_A \frac{(A-2Z)^2}{A}$$

Similarly for daughter nuclei,

$$M\left(\frac{A}{2},\frac{Z}{2}\right)c^2 = \frac{Z}{2}M_Hc^2 + \frac{N}{2}m_nc^2 - a_V \frac{A}{2} + a_S \left(\frac{A}{2}\right)^{\frac{2}{3}} + a_C \frac{(Z/2)^2}{(A/2)^{\frac{1}{3}}} + a_A \frac{(A-2Z)^2}{2A}$$

$$\therefore Q = M(A,Z) - 2 \times M\left(\frac{A}{2},\frac{Z}{2}\right)$$

$$= a_S A^{\frac{2}{3}}\left(1 - \frac{2}{2^{2/3}}\right) + a_C \frac{Z^2}{A^{\frac{1}{3}}}\left(1 - \frac{2^{1/3}}{2}\right)$$

$$= -0.26\, a_S\, A^{\frac{2}{3}} + 0.37\, a_C \frac{Z^2}{A^{\frac{1}{3}}}$$

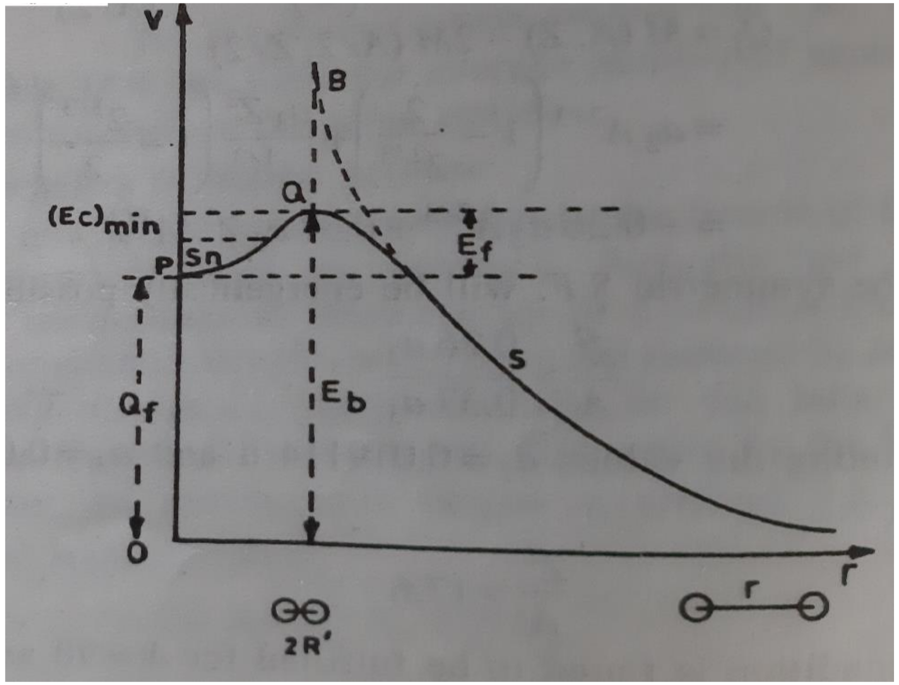
Thus the symmetric spontaneous nuclear fission is energetically possible $(Q > 0)$ if

$$\frac{Z^2}{A} > \frac{0.26\, a_S}{0.37\, a_C}$$

Substituting the values of a_C, a_S we get,

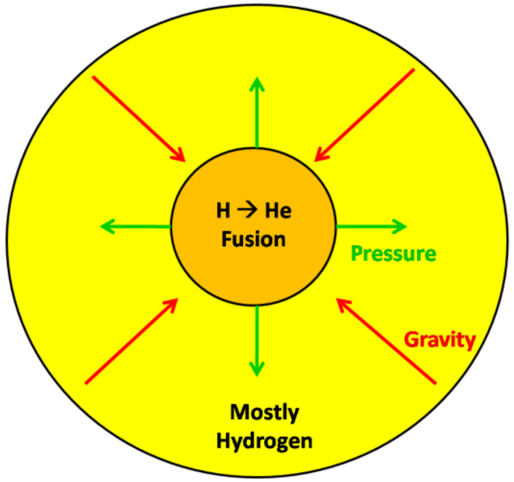
$$\frac{Z^2}{A} > 17.2$$

- The SF condition is satisfied for $A > 90; Z > 40$. (For $A > 90; Z > 40, \frac{Z^2}{A} = 17.6$).
- Thus for $A > 90$ SF is energetically possible. But in reality is rarely possible due to quantum mechanical barrier penetration problem.
- Even for the heaviest atomic nuclei (U-238), it is very rarely observed. About 1 SF per hour in 1gm of U-238 corresponding to half-life 2×10^{17} years.
- The reason is Coulomb barrier which is very large here compared to the $\alpha - decay$ case. Here activation energy (difference between Coulomb barrier and Q-value) is needed in the form of neutron kinetic energy for the fission to occur. For $A > 250$, there is no need for activation energy, but in this case the heavy nucleus undergoes instantaneous spontaneous fission.



NUCLEAR MODELS(L4)

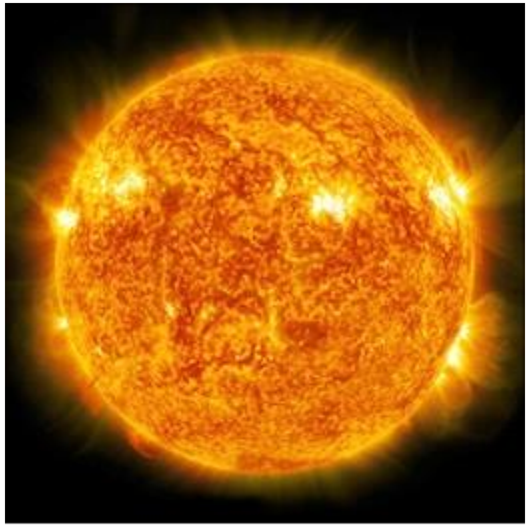
Formation of Neutron Stars



How is a Star formed?

- After Big Bang universe is mainly filled up with $H - atoms$(75%) and He-atoms (25%). H-atoms coalesce due to gravity acting inward and start collapsing.
- These $H - atoms$ are the main ingredients of a star.
- As $H - atoms$ start coming closer their gravitational potential energy converts into kinetic energy and become hotter and hotter thereby increasing the temperature.
- Eventually the temperature becomes so large (about millions °K) that a plasma state of hot gas of *H-nuclei* and *free electrons* was formed. Actually at that very high temperature electrons were stripped off *H-atoms* leaving behind only protons as *H-nuclei*. A big dense ball of protons is obtained. PROTOSTAR
- At such high temperature (10 millions °K) nuclear fusion starts operating as the density increases further and two *H-nuclei* combine to form *He* nucleus with evolution of heat and light energy.
- Depending upon the size and mass of the star this fusion process can continue all the way up to *Fe* formation.

Formation of Neutron Stars



How is a Neutron Star formed?

- The gravitational inward pulling is held up by the outward nuclear radiation pressure and as long as this condition is maintained the star exits.
- Once this nuclear fusion stops, the gravitational inward pull dominates and the outer layer starts shrinking and consequently the density and temperature at the core becomes large. The outer layers get rebounded by the highly dense inner core and Supernovae explosion takes place leaving behind a dense core. The star ends its life.
- Depending upon the size and mass of the star the inner hard core can further shrink due to gravitational inward pull. Consequently the density and temperature becomes huge and enormous pressure builds up.
- As a result the electron in the outer shell of H- atom falls on inner proton and gets combined to produce neutrons and neutrinos. Neutrinos being non-interacting escape the core and roam around the interstellar space and constantly hitting on earth surface. Due to this enormous pressure all protons get converted to neutrons and now we obtain a dense ball of neutrons. This is called neutron star. Although there could be a little number of protons on the outer edge of the star.

- The neutron star is a highly dense hot object. One tea spoon of neutron star is weighing $5\times 10^{12}\ kg$.
- Neutron stars are of (10-20) km of radius which is as big as a standard city is. However the mass is (2-10) times the mass of the Sun.
- A neutron star has approximately 10^{56} number of neutrons.

How could so many neutrons form a stable neutron stars whereas two neutrons or 4 neutrons can not form a stable nucleus?

- Gravitational force is much weaker than the Coulomb force.
- Actually gravitational force is much weaker than nuclear force at the nuclear scale.
- At the astronomical scale we can not neglect gravitational force when so many neutrons are packed inside a small volume.

Is Bethe-Weizsacker (BW) mass formula valid for celestial/heavenly bodies?

Remember: The BW formula was derived on the basis of observations on terrestrial or laboratory observations of the nuclei.

Since we are considering celestial bodies the gravitational potential energy of the neutron star can not be neglected. For a mass of M, the GPE is $\frac{3}{5}\frac{GM^2}{R}$

The BE expression would be:

$$B = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(N-Z)^2}{A} + \delta + \frac{3}{5}\frac{GM^2}{R}$$

- Consider A number of neutrons and each has mass m_n. Therefore, mass of the neutron star is $M = Am_n$
- Coulomb energy term (red in colour) vanishes as neutrons are neutral and Pairing energy term (red in colour) is neglected. The GPE makes a positive contribution, hence BE would increase.

$$B = a_V A - a_A A + \frac{3}{5}\frac{G m_n^2 A^{\frac{5}{3}}}{R_0}$$

- As volume increases the contribution of volume term dominates over the surface term $(\sim \frac{1}{R})$ in the BE formula.
- Hence, the condition for stable neutron star is $B \geq 0$

$$B = (a_V - a_A)A + \frac{3}{5}\frac{Gm_n^2 A^{\frac{5}{3}}}{R_0} \geq 0$$

Since $(a_V - a_A)$ is negative, that means stable neutron star would not have been formed unless the GPE term is considered.

Therefore,

$$-7.9\ MeV \geq -\frac{3}{5}\frac{Gm_n^2 A^{\frac{2}{3}}}{R_0}$$

$$\frac{3}{5}\frac{Gm_n^2 A^{\frac{2}{3}}}{R_0} \geq 7.9\ MeV$$

$$A^{\frac{2}{3}} \geq \frac{5R_0}{3Gm_n^2} \times 7.9\ MeV$$

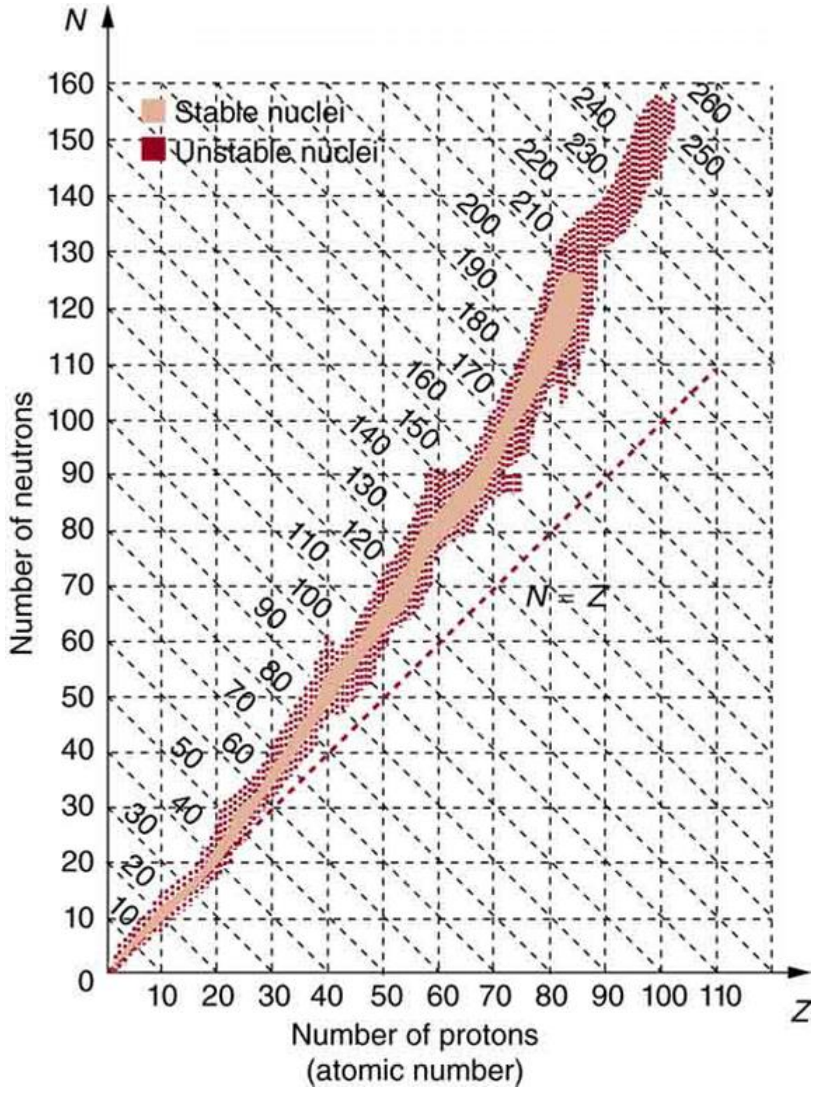
$$A^{\frac{2}{3}} \geq \frac{5 \times 1.2 \times 10^{-15}}{3 \times 6.67 \times 10^{-11} \times (1.67 \times 10^{-27})^2} \times 7.9 \times 1.6 \times 10^{-13}$$

$$A^{\frac{2}{3}} \geq 1.4 \times 10^{37} \approx 10^{37} \qquad \Rightarrow A \approx 10^{56}$$

Hence mass of neutron star = $1.67 \times 10^{-27} \times 10^{56} = 0.1 \times M_S$ since $M_S = 10^{30} Kg$

Conclusion: Hence Semi-empirical mass formula is valid for even celestial bodies (having radius (10-20)Km) and it can estimate the mass of a neutron star to a good approximation.

- From $N - Z$ plot it is clear that nuclei belonging to solid light brown regions are stable against $\beta - decay$.
- There are points in this region for large mass number A which are stable against $\alpha - decay$ only.
- The dotted maroon points (except a few ones at the top edge) denote unstable nuclei which become stable against $\beta - decay$ or $\alpha - decay$.
- For heavy nuclei nuclear fission can occur and stable nuclei of lighter A are formed.
- No nuclei will exist beyond $\mathrm{Z} = 100$ and $N = 160$ (around, not exact).
- Those nuclei formed will instantaneously be broken up
- So for fixed value of N there is limit on choice of Z and vice e versa.



What is Neutron Drip line?

Conversion of neutrons to protons is energetically favourable.

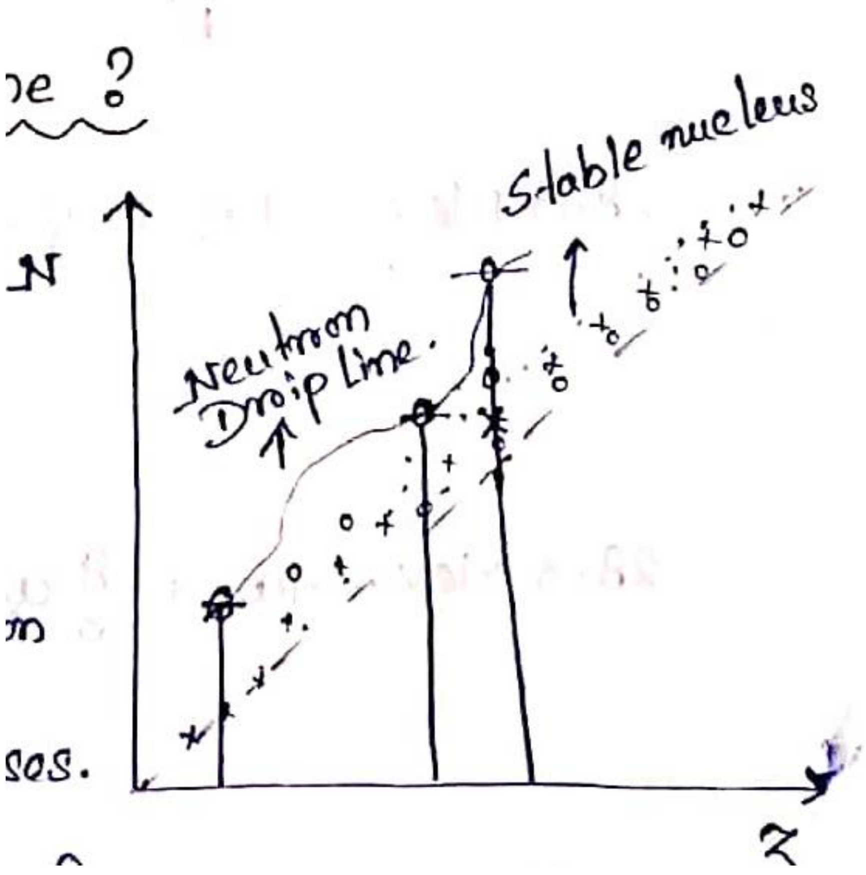
As it happens the number of neutrons decrease in a nucleus.

Beyond NDL no nucleus will be formed that is neutrons will drip

Out of the nucleus if one tries to put it inside.

How far is this decrement possible ?

Neutron Separation Energy (NSE): The minimum amount of energy needed by a nucleon to come out of a nucleus and to become completely isolated is known as the NSE.



For a fixed value of Z , the number of neutrons that can be added to a nucleus is limited. If we join all these points on $N-Z$ plot for different Z values, the drawn curve would represent the NDL.

$$ {}^{A}_{Z}X + {}^{1}_{0}n \rightarrow {}^{A+1}_{\;\;\;Z}X $$

$$ m(A,Z)c^2 + m_n c^2 > m(A+1,Z)c^2 $$

This reaction is spontaneously possible when mass of parent nuclei is greater than that of daughter nuclei

$$ {}^{A}_{Z}X \rightarrow {}^{A-1}_{Z}X + {}^{1}_{0}n $$

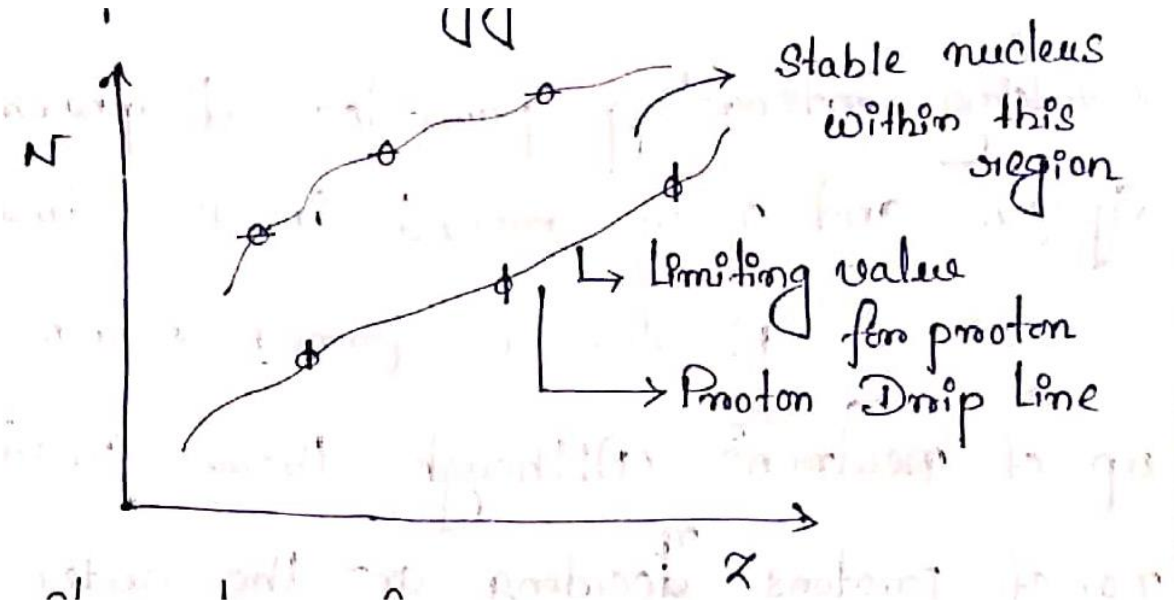
The NSE is :

$$ S_n = m(A-1, Z)c^2 + m_n c^2 - m(A, Z)c^2 $$

What is Proton Drip line?

For a fixed number of N, number of protons that can be added to a nucleus is limited.

If we join all these points for different N values, the line obtained is would be called Proton Drip line (PDL).



Proton Separation Energy (PSE):

The minimum amount of energy required for a proton to become completely free from a nucleus is called PSE.

In $N - Z$ plot only the nuclei lying between NDP and PDL can be stable.

Successes of Liquid Drop Model:

- Macroscopic properties of the nuclei can be explained.
- Collective motion of nucleons are considered. Nucleons are constantly in motion within the nucleus.
- BE curve can nicely be explained by this model except a few issues.
- Various nuclear phenomena like Stability of a nucleus, Fission, radioactive decays, neutron drip line, proton drip line can also be explained.
- Semi empirical mass formula can limit the stability of nuclei against $\alpha - decay$ and nuclear fission

Limitations of the model:

- Nuclei containing magic number (2,4,8,16,20,28,50,82,126) of nucleons exhibit high stability. This typical feature of nuclei can not be explained.
- BE of nuclei having mass number below 20 and above 180 is difficult to be explained by this model.
- According to Liquid drop model energy levels are closely spaced. But low lying excited states are indeed widely space as observed by the experiments, which is contrary to the prediction of Liquid Drop Model.

Mass Parabolas – Odd A

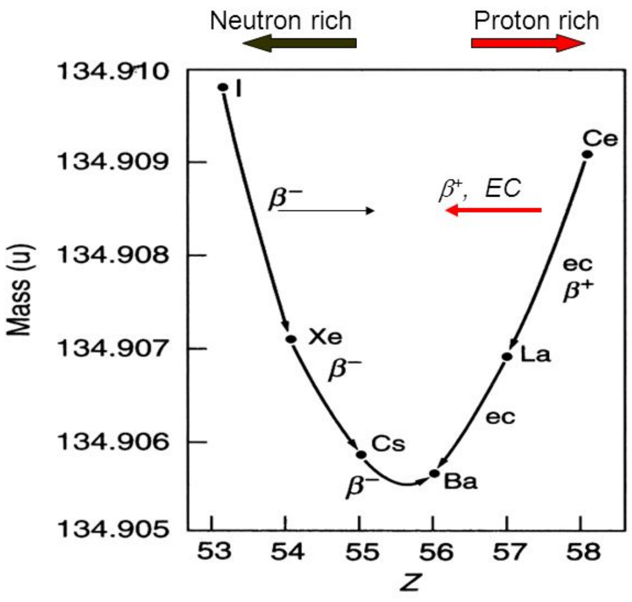
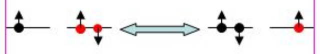


Fig 4.3 Mass parabola for A=135 showing **One Stable Nuclide with Z=56**

We can find from the SEMF mass parabola an equation for the minimum of the $M_A(Z)$ curve

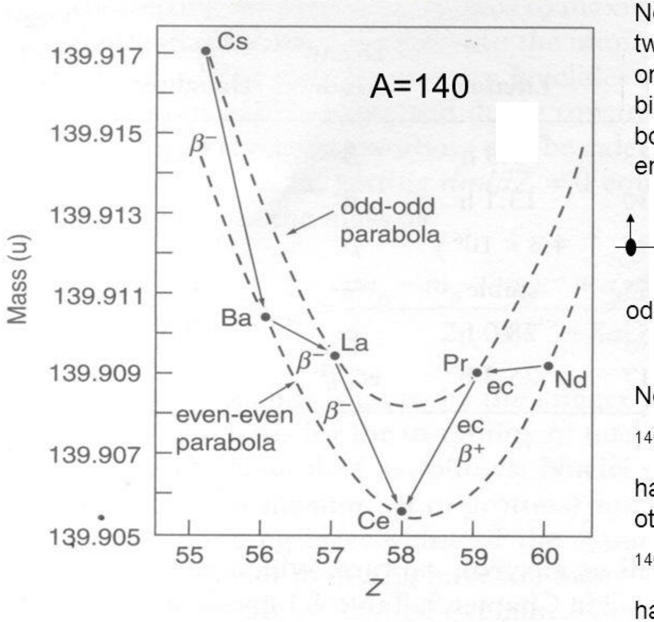


Physically one is always having tight binding on either the neutron side or the proton side of the nucleus.

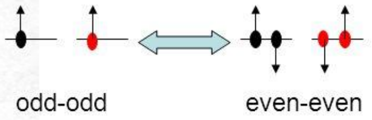
- For disappearance of Pairing energy term only one parabola exists and for $odd - A$ nuclei only one stable nucleus exist for each isobar.
- From semi empirical mass formula we can draw mass parabola for $odd - A$ cases against Z. As one crawls down the left limb of the parabola, the nuclei become stable against β^- decay since in this case proton number (Z) increases by one unit $(n \rightarrow p + \beta^- + \bar{\nu}\ )$.
- Similarly if one crawls down from right limb of the parabola, the nuclei become stable against β^+ decay since in this case proton number (Z) decreases by one unit $(p \rightarrow n + \beta^+ + \nu)$.
- Here only one stable nuclei exist, i.e., $Z = 56$ for $\mathrm{A} = 135$.

Mass Parabolas - even A

(looking down the valley of stability – A decreasing)



Note that for even A there exist two mass parabolas – the top one for low pairing energy binding (ODD-ODD) and the bottom one for high binding energy (EVEN-EVEN)



Note that some decay such as

$^{140}Nd \rightarrow ^{140}Pr$

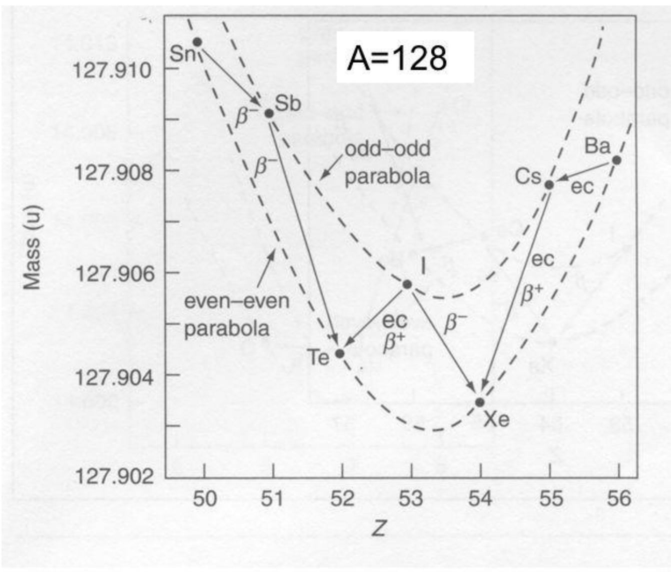
have quite low Q energy, while other such as

$^{140}Pr \rightarrow ^{140}Ce$

have large Q energy

Mass Parabolas - even A

(looking down the valley of stability – A decreasing)



Sometimes the positioning of the isobars is such that one can get TWO STABLE ISOBARS

Eg. ^{128}**Te** and ^{128}**Xe**

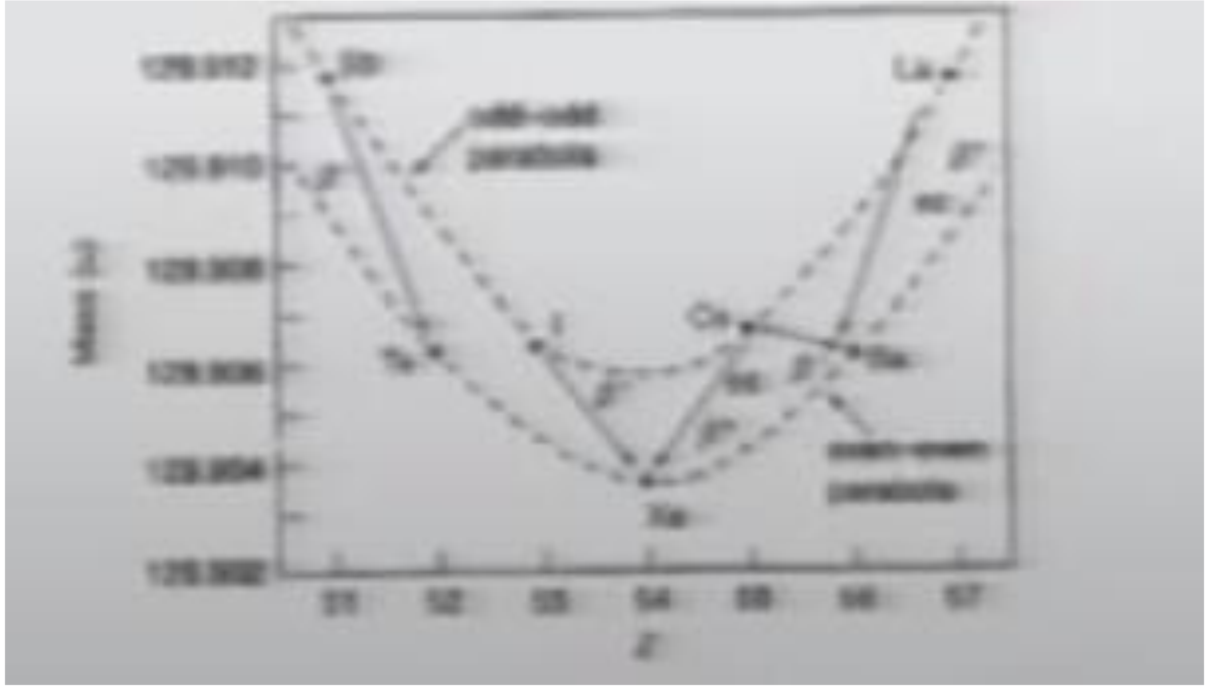
and the strange phenomenon that a nuclide such as ^{128} **I** can both β^+ and β^- decay!

The displacement of the parabolas is of course

$$2\delta = \frac{2a_P}{A^{3/4}}$$

- For $even - A$ nuclei, two parabolas can be drawn for odd-odd and even-even nuclei. This happens due to presence of Pairing energy term.
- The odd-odd parabola would lie above the even-even parabola and the shifting would be measured by pairing coefficient.
- For $even - A$ nuclei there might be one, two or three stable nuclei depending upon masses which can be understood by semi-empirical mass formula.

- For $A = 140$ there will be one stable nuclei at $A = 58$. This can be explained by semi empirical mass formula.
- For $A = 128$ there will be two stable nuclei corresponding to $A = 52{,}54$.
- For odd-odd parabola there is one minimum mass corresponds to Iodine $(I, Z = 53)$. For the even-even parabola there are two nuclei $(Te, Z = 52; Xe, Z = 54)$.
- Iodine (I) can be transformed into Te or Xe either by β^- or β^+ decay. Here Te and Xe are considered as stable nuclei since transformation from Te to Xe can only possible via double-β decay which has yet not been observed.
- For $A = 130$ there will be three stable nuclei at $A = 52{,}54{,}56$.
- For even-even parabola *Te (52)* can not go to *I (53)* because *I* is situated slightly higher than *Te* and moreover, *I* can not go to *Xe* as before (*A=128*).
- I can decay into Xe through $\beta^- - decay$.
- *Ba* can not decay into *Cs* as the later is at slightly higher state. So *Ba* is stable .



List of Books of Nuclear Physics:

1. Introductory Nuclear Physics: K.S. Krane
2. Introductory Nuclear Physics: Samuel.S.M.Wong
3. Theoretical Nuclear Physics: Blatt-Weiskopf
4. Nuclear Physics: S.N.Ghsohal
5. Physics of the Nucleus: M.A.Preston
6. Basic Ideas and Concepts in Nuclear Phsyics: Heyde
7. Nuclear Physics: Theory and Experiment: Roy & Nigam